P 159 - 161

2 (a) $f(z) = \frac{1}{3z^2+1}$

f is differentiable except $z = \pm \frac{\sqrt{3}}{3} i$.

Then f is analytic in the region bounded by C_1, C_2.

Hence, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

(b) $f(z) = \frac{z+2}{\sin(\frac{z}{2})}$

f is differentiable except $z = 2n\pi$, $n \in \mathbb{Z}$

Then f is analytic in the region bounded by C_1, C_2.

Hence, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

(c) $f(z) = \frac{z}{1-e^z}$

f is differentiable except $z = 2n\pi i$, $n \in \mathbb{Z}$

Then f is analytic in the region bounded by C_1, C_2.

Hence, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

$\square$

3 Let $z_0 = 2+i$ and $f(z) = (z-z_0)^{n-1}$

We can take R large enough s.t. the given reetangle is contained in $B(x_0, R)$.

Let $C_0 := \partial B(x_0, R)$.

Note that f is differentiable everywhere when $n \geq 1$.

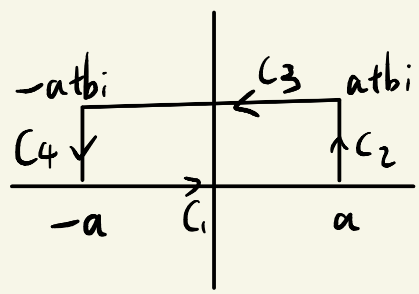
Otherwise, f is differentiable except $z = z_0$.

Therefore, f is analytic in the region bounded by C, C_0.

Hence $\int_C (z-2-i)^{n-1} dz = \int_{C_0} (z-z_0)^{n-1} dz = \begin{cases} 0, & \text{when } n = \pm 1, \pm 2, \dots \\ 2\pi i, & \text{when } n = 0 \end{cases}$.

$\square$

4 (a)



Let $f(z) = e^{-z^2}$.

$$\int_{C_1} f(z)\,dz = \int_{-a}^{a} e^{-x^2} dx = 2\int_0^a e^{-x^2} dx$$

$$\int_{C_2} f(z)\,dz = \int_0^b e^{-(a+iy)^2} d(a+iy) = i e^{-a^2} \int_0^b e^{y^2} e^{-2aiy} dy$$

$$\int_{C_3} f(z)\,dz = \int_a^{-a} e^{-(x+bi)^2} dx = -e^{b^2} \int_{-a}^{a} e^{-x^2} e^{-2bix} dx$$

$$= -2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

$$\int_{C_4} f(z)\,dz = \int_b^0 e^{-(-a+iy)^2}\,d(-a+iy) = -ie^{-a^2}\int_0^b e^{y^2}e^{2aiy}\,dy$$

By Cauchy-Goursat, $\int_{C_1+C_2+C_3+C_4} f(z)\,dz = 0$.

Note that $\int_{C_2+C_4} f(z)\,dz = ie^{-a^2}\int_0^b e^{y} e^{-2aiy} - e^{2aiy}\,dy$

$$= 2e^{-a^2}\int_0^b e^{y}\sin(2ay)\,dy.$$

Then $0 = \int_{C_1+C_2+C_3+C_4} f(z)\,dz = 2\Big(\int_0^a e^{-x^2}dx - e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$

$$+ e^{-a^2}\int_0^b e^{y}\sin(2ay)\,dy.$$

Hence, $\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y}\sin 2ay\,dy.$

(b) Note that $\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b \left|e^{y^2}\sin 2ay\right|dy = \int_0^b e^{y^2}dy$

and $\lim_{a\to\infty} e^{-(a^2+b^2)} = 0$.

Therefore $\lim_{a\to\infty} e^{-(a^2+b^2)}\int_0^b e^{y}\sin 2ay\,dy = 0.$

Hence $\int_0^\infty e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$

$\square$

5 By Cauchy-Goursat, if f is entire, then

$$\int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz$$

$$\int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz$$

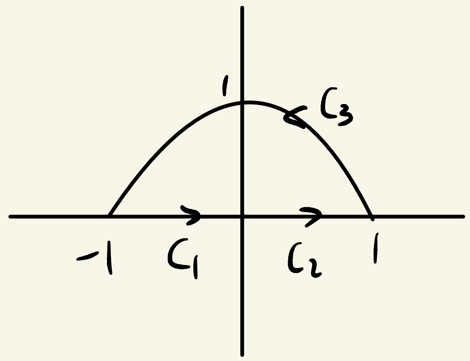
Then $$\int_C f(z)\,dz = \int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= \int_{C_3} f(z)\,dz - \int_{C_3} f(z)\,dz$$

$$= 0$$

$\square$

6



$$\int_{C_1} f(z)\,dz = \int_1^0 \sqrt{r}\,e^{\frac{\pi}{2}i}\,d(-r) = \frac{2}{3}i$$

$$\int_{C_2} f(z)\,dz = \int_0^1 \sqrt{r}\,dr = \frac{2}{3}$$

$$\int_{C_3} f(z)\,dz = \int_0^{\pi} e^{\frac{i}{2}\theta}\, i e^{i\theta} d\theta = i\int_0^{\pi} e^{\frac{3}{2}i\theta} d\theta$$

$$= \frac{2}{3}\left(e^{\frac{3}{2}\pi i} - 1\right)$$

$$= -\frac{2}{3}(i+1)$$

$$\int_C f(z)\,dz = \int_{C_1+C_2+C_3} f(z)\,dz = \frac{2}{3}i + \frac{2}{3} - \frac{2}{3}(i+1) = 0$$

We cannot apply Cauchy - Goursat since f is not analytic at $z=0$.

P 170 - 172

1.(a) $\displaystyle\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}dz = 2\pi i\, e^{-\frac{\pi}{2}i} = 2\pi$

(b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}dz = 2\pi i\,\frac{\cos 0}{0+8} = \frac{\pi}{4}i$

(c) $\displaystyle\int_C \frac{z}{2z+1}\,dz = 2\pi i\,\frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{2}\pi i$

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz = \frac{2\pi i}{3!}\left(\frac{d^3}{dz^3}\cosh z\right)\Big|_{z=0} = \frac{\pi i}{3}\sinh(0) = 0$

(e) $\displaystyle\int_C \frac{\tan\frac{z}{2}}{(z-x_0)^2} = 2\pi i\left(\frac{d}{dz}\tan\frac{z}{2}\right)\Big|_{z=x_0} = \pi i\sec^2(x_0/2)$

$\square$

2. Let $C = \{z \in \mathbb{C} : |z-i| = 2\}$

(a) $\displaystyle\int_C \frac{1}{z^2+4}\,dz = 2\pi i\,\frac{1}{2i+2i} = \frac{\pi}{2}$

(b) $\displaystyle\int_C \frac{1}{(z^2+4)^2}\,dz = 2\pi i\left(\frac{d}{dz}\frac{1}{(z+2i)^2}\right)_{z=2i} = 2\pi i(-2)\frac{1}{(2i+2i)^3}$

$\displaystyle= \frac{\pi}{16}$

$\square$

3 $$g(2) = \int_C \frac{2s^2 - s - 2}{s - 2} ds = 2\pi i\,(2\times 2^2 - 2 - 2) = 8\pi i$$

When $|z| > 3$, $\dfrac{2s^2 - s - z}{s - z}$ is analytic inside and on C. Applying Cauchy-Goursat, we have $g(z) = 0$.

$\square$

4 When z is inside C,

$$g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} ds = \frac{2\pi i}{2!}\left(\frac{d}{ds}(s^3 + 2s)\Big|_{s=z}\right)$$

$$= 6\pi i z$$

When z is outside C, $\dfrac{s^3 + 2s}{(s - z)^3}$ is analytic inside and on C. Applying Cauchy-Goursat, $g(z) = 0$.

$\square$

10. By Cauchy's inequality,

$$|f''(z)| \leq \frac{2}{R^2} \max_{z \in C_R} |f(z)| \qquad C_R = \partial B_R(z)$$

$$\leq \frac{2}{R^2} A(R + |z|) \quad (\text{by assumption})$$

$$= 2A \frac{R+|z|}{R^2} \to 0 \qquad \text{as } R \to \infty.$$

Therefore $f''(z) \equiv 0$.

Then $f'(z) \equiv a_1$ for some constant a_1

Then $f(z) = a_1 z + a_0$.

Since $|f(0)| = |a_0|$, by assumption, $|a_0| \leq A|0| = 0$.

Then $a_0 = 0$

Hence, $f(z) = a_1 z$.

$\square$